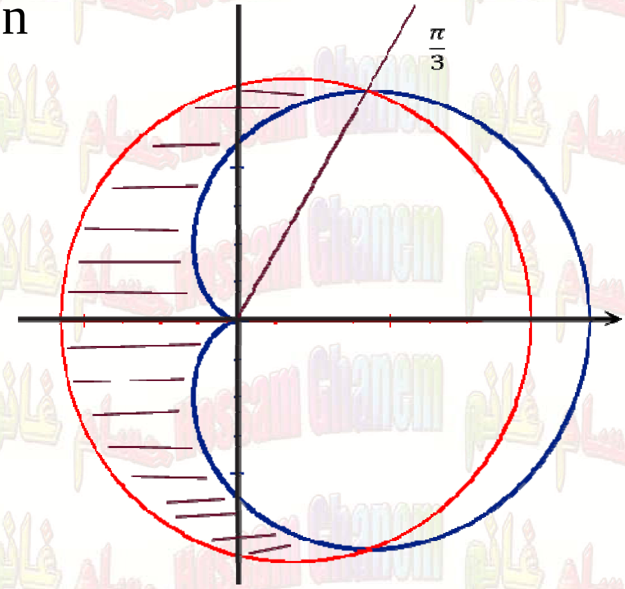
(41) 11.4 Areas In Polar Coordinates

Example 1	Find the area of the region that is outside the graph of $r = 3(1+\cos\theta)$ and inside the graph of $r = 4+\cos\theta$	16 Dec. 1999

Solution



Intersection point

$3(1+\cos\theta) = 4+\cos\theta$

$3+3\cos\theta = 4+\cos\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$$A = \frac{1}{2}\int_a^b r^2\,d\theta$$

$$A = \frac{1}{2}\cdot 2\int_{\frac{\pi}{3}}^{\pi}(4+\cos\theta)^2\,d\theta - \int_{\frac{\pi}{3}}^{\pi}[3(1+\cos\theta)]^2\,d\theta$$

$$= \int_{\frac{\pi}{3}}^{\pi}[16+8\cos\theta+\cos^2\theta-9-54\cos\theta-9\cos^2\theta]\,d\theta$$

$$= \int_{\frac{\pi}{3}}^{\pi}(25-46\cos\theta-8\cos^2\theta)\,d\theta = \int_{\frac{\pi}{3}}^{\pi}\big(25-46\cos\theta-4(1+\cos 2\theta)\big)\,d\theta$$

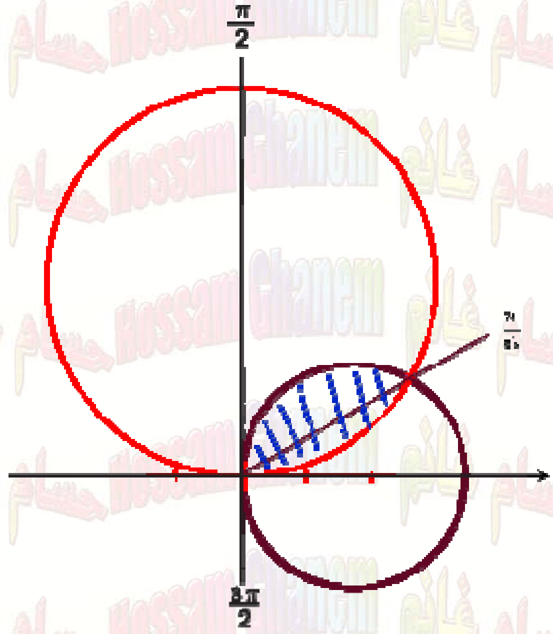
$$= \int_{\frac{\pi}{3}}^{\pi}(25-46\cos\theta-4-4\cos 2\theta)\,d\theta = \int_{\frac{\pi}{3}}^{\pi}(21-46\cos\theta-4\cos 2\theta)\,d\theta$$

$$= \Big[21\theta-46\sin\theta-2\sin 2\theta\Big]_{\frac{\pi}{3}}^{\pi}$$

$$= 21\pi-0-0-\left(7\pi-\frac{46\sqrt{3}}{2}-\frac{2\sqrt{3}}{2}\right) = 14\pi+24\sqrt{3}$$

Example 2 Find the area of the region that is inside the graphs of both equations $r = \sqrt{3}\sin\theta$ and $r = \cos\theta$ 15 May 1999

Solution



Intersection point

$$\sqrt{3}\sin\theta = \cos\theta$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

$$A = \frac{1}{2}\int_a^b r^2 \, d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{6}} \left(\sqrt{3}\sin\theta\right)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta \; d\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{6}} \frac{1}{2}(1-\cos 2\theta)\, d\theta + \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$$

$$= \frac{3}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} + \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{3}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4} - (0)\right] + \frac{1}{4}\left[\frac{\pi}{2} + 0 - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)\right]$$

$$= \frac{3\pi}{24} - \frac{3\sqrt{3}}{16} + \frac{\pi}{8} - \frac{\pi}{24} - \frac{\sqrt{3}}{16}$$

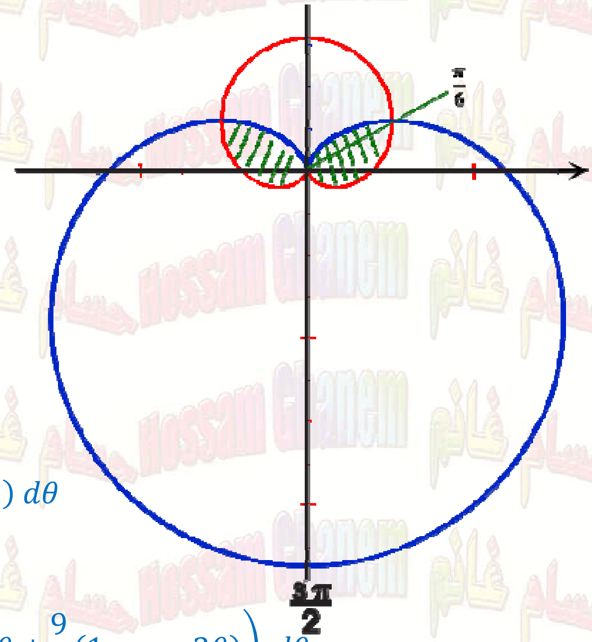
$$= \left(\frac{3+3-1}{24}\right)\pi - \frac{3-1}{16}\left(\sqrt{3}\right) = \frac{5\pi}{24} - \frac{\sqrt{3}}{8}$$

Example 3 Sketch the graphs of the polar equations $r = 3 - 3\sin\theta$ and $r = 1 + \sin\theta$ and find the area of the region that lies inside both graphs. 31 August 2008 A

Solution

Intersection point

$$3 - 3\sin\theta = 1 + \sin\theta$$
$$4\sin\theta = 2$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}$$



$$A = \int_a^b r^2 d\theta$$

$$A = 2\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1+\sin\theta)^2\, d\theta + 2\cdot\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3-3\sin\theta)^2\, d\theta$$

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1 + 2\sin\theta + \sin^2\theta)\, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (9 - 18\sin\theta + 9\sin^2\theta)\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(1 + 2\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right) d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(9 - 18\sin\theta + \frac{9}{2}(1-\cos 2\theta)\right) d\theta$$

$$= \left[\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}} + \left[9\theta + 18\cos\theta + \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{6} - \frac{2\sqrt{3}}{2} + \frac{\pi}{12} - \frac{\sqrt{3}}{8} - \left(-\frac{\pi}{2} - 0 - \frac{\pi}{4} - 0\right) + \left[\frac{9\pi}{2} + 0 + \frac{9\pi}{4} - 0 - \left(\frac{9\pi}{6} + \frac{18\sqrt{3}}{2} + \frac{9\pi}{12} + \frac{9\sqrt{3}}{8}\right)\right]$$

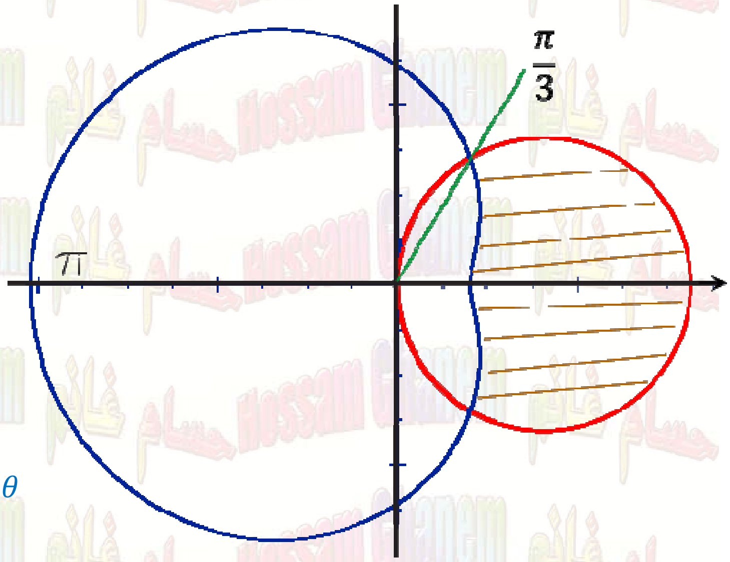
$$= \left(\frac{1}{6} + \frac{1}{12} + \frac{1}{2} + \frac{1}{4} + \frac{9}{2} + \frac{9}{4} - \frac{9}{6} - \frac{9}{12}\right)\pi + \left(\frac{-2\sqrt{3}}{2} - \frac{\sqrt{3}}{8} - \frac{18\sqrt{3}}{2} - \frac{9\sqrt{3}}{8}\right)$$

$$= \left(\frac{2+1+6+3+54+27-18-9}{12}\right)\pi + \left(\frac{-8-1-72-9}{8}\right)\sqrt{3}$$

$$= \frac{66}{12}\pi - \frac{90\sqrt{3}}{8} = \frac{11}{2}\pi - \frac{90\sqrt{3}}{8}$$

Example 4 Find the area of the region that is inside the circle $r = 4\cos\theta$ and outside the limacon $r = 3 - 2\cos\theta$. 14 Dec. 1998

Solution



Intersection point

$$4\cos\theta = 3 - 2\cos\theta.$$
$$6\cos\theta = 3$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}$$

$$A = \frac{1}{2}\int_a^b r^2\,d\theta$$

$$A = 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}}(16\cos^2\theta)\,d\theta - 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}}(3-2\cos\theta.)^2\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} 8(1+\cos 2\theta)\,d\theta - \int_0^{\frac{\pi}{3}}(9 - 12\cos\theta + 4\cos^2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{3}}(8 + 8\cos 2\theta)\,d\theta - \int_0^{\frac{\pi}{3}}(9 - 12\cos\theta + 2 + 2\cos 2\theta)\,d\theta$$

$$= \Big[8\theta + 4\sin 2\theta\Big]_0^{\frac{\pi}{3}} - \Big[11\theta - 12\sin\theta + \sin 2\theta\Big]_0^{\frac{\pi}{3}}$$

$$= \frac{8\pi}{3} + \frac{4\sqrt{3}}{2} - 0 - \left[\frac{11\pi}{3} - \frac{12\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - 0\right]$$

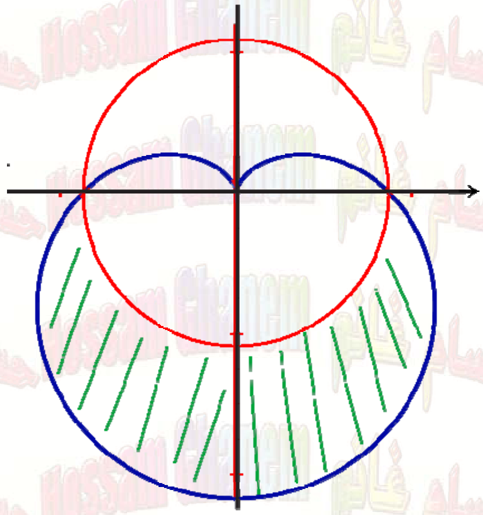
$$= \frac{8\pi}{3} + 2\sqrt{3} - \frac{11\pi}{3} + 6\sqrt{3} - \frac{\sqrt{3}}{2} = \left(\frac{8-11}{3}\right)\pi + \left(\frac{4+12-1}{2}\right)\sqrt{3} = \frac{15\sqrt{3}}{2} - \pi$$

Example 5

41 14 January 2012

Find the area of the region that lies inside the polar curve $r = 1 - \sin\theta$ and outside the polar curve $r = 1$. (5 pts)

Solution



$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$A = 2\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{0}(1-\sin\theta)^2 d\theta - 2\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{0}(1)^2 d\theta$$

$$A = \int_{-\frac{\pi}{2}}^{0}(1 - 2\sin\theta + \sin^2\theta)d\theta - \int_{-\frac{\pi}{2}}^{0} d\theta$$

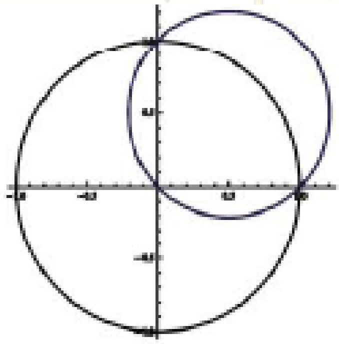
$$A = \int_{-\frac{\pi}{2}}^{0}\left(1 - 2\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right)d\theta - \int_{-\frac{\pi}{2}}^{0} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0}\left(1 - 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta - 1\right)d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0}\left(\frac{1}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \left[\frac{1}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{0} = 0 + 2 - 0 - \left(-\frac{\pi}{4} + 0 - 0\right) = 2 + \frac{\pi}{4} = \frac{\pi + 8}{4}$$

Homework

1	(2 + 2 pts.) (a) Sketch the polar curves $r = 2 - 2\cos\theta$ and $r = 2$ (b) Set up (but do not evaluate) the integral(s) representing the area inside the curves $r = 2 - 2\cos\theta$ and $r = 2$	36 June 6, 2010
2	33 June 2009 A The graphs of the polar equations $r = \sin\theta + \cos\theta$ and $r = 1$ are shown. Find the area of the region inside both circles	

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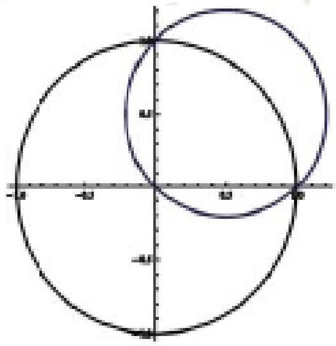
33 June 2009 A

The graphs of the polar equations

$r = \sin\theta + \cos\theta$ and $r = 1$

are shown.

Find the area of the region inside both circles



Solution

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$A = \frac{1}{4}\pi(1)^2 + 2\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{0}(\sin\theta + \cos\theta)^2\, d\theta$$

$$= \frac{\pi}{4} + \int_{-\frac{\pi}{2}}^{0}(\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{\pi}{4} + \int_{-\frac{\pi}{2}}^{0}(1 + 2\sin\theta\cos\theta)\, d\theta$$

$$= \frac{\pi}{4} + \int_{-\frac{\pi}{2}}^{0}(1 + \sin 2\theta)\, d\theta$$

$$= \frac{\pi}{4} + \left[\theta - \frac{1}{2}\cos 2\theta\right]_{-\frac{\pi}{2}}^{0}$$

$$= \frac{\pi}{4} - \frac{1}{2} - \left(-\frac{\pi}{2} + \frac{1}{2}\right) = \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{2} - \frac{1}{2} = \frac{3\pi}{4} - 1 = \frac{3\pi - 4}{4}$$